THE PRODUCT TOPOLOGY

EXAMPLE: Suppose $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are topological spaces. Let $\mathcal{B} = \{T \times U : T \in \mathcal{T} \text{ and } U \in \mathcal{U}\}$.

- Show that, in general, $\mathcal{B}$ is **not** a topology on $X \times Y$.
- Prove that $\mathcal{B}$ **is** a **base** for a topology on $X \times Y$.

DEFINITION: Suppose $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are topological spaces. Let $\mathcal{B} = \{T \times U : T \in \mathcal{T} \text{ and } U \in \mathcal{U}\}$.

The topology generated by $\mathcal{B}$ is called the **product topology** on $X \times Y$.

EXAMPLE: Suppose $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are topological spaces and $\mathcal{B}$ is a base for $\mathcal{T}$ and $\mathcal{D}$ is a base for $\mathcal{U}$.

Show $\{B \times D : B \in \mathcal{B},\ D \in \mathcal{D}\}$ is a base for the product topology on $X \times Y$.

EXAMPLE: Let $\mathbb{R}$ be the real numbers. Sketch some basic open sets for the product topology:

- $\mathbb{R}_{\mathcal{E}} \times \mathbb{R}_{\mathcal{E}}$ where $\mathbb{R}_{\mathcal{E}}$ denotes $\mathbb{R}$ with the Euclidean topology.
- $\mathbb{R}_{\mathcal{S}} \times \mathbb{R}_{\mathcal{S}}$ where $\mathbb{R}_{\mathcal{S}}$ denotes $\mathbb{R}$ with the Sorgenfrey topology.
- $\mathbb{R}_{\mathcal{E}} \times \mathbb{R}_{\mathcal{S}}$
- $\mathbb{R}_{\mathcal{S}} \times \mathbb{R}_{\mathcal{E}}$

Let $Q = \{(x, y) : x \geq 0,\ y \geq 0\}$. Under which of the product topologies above is Q open? Closed? Clopen?

EXAMPLE: Suppose X and Y are nonempty sets.

Define $\pi_X : X \times Y \to X$ by $\pi_X(x, y) = x$ and $\pi_Y : X \times Y \to Y$ by $\pi_Y(x, y) = y$.

Prove π_X and π_Y are surjective functions.

NOTE: π_X and π_Y are called the **projections** of $X \times Y$ onto X and Y, respectively.

EXAMPLE: Consider Let $X = \mathbb{R}$ and $Y = [0, \infty)$.

- Sketch $X \times Y$.
- Let G be the graph of $y = 6$. Find $\pi_X(G)$ and $\pi_Y(G)$.
- Sketch $\pi_X^{-1}(\{3\})$ and $\pi_Y^{-1}(\{3\})$
- Find $\pi_Y\left(\pi_X^{-1}(\{3\})\right)$.

EXAMPLE: Suppose $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are topological spaces and $X \times Y$ has the product topology.

- Prove π_X and π_Y are continuous.
- Show the product topology is the **weakest** topology on $X \times Y$ for which both projections are continuous.
- Prove π_X and π_Y are open.

THEOREM: Suppose $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are topological spaces and let $X \times Y$ have the product topology.

$$X \times Y \text{ is connected if and only if } X \text{ and } Y \text{ are connected.}$$

THEOREM: Suppose $F : (Z, \mathcal{W}) \to (X, \mathcal{T})$ and $G : (Z, \mathcal{W}) \to (Y, \mathcal{U})$ are continuous.

Then there exists a unique continuous function $H : (Z, \mathcal{W}) \to (X \times Y, \mathcal{T} \times \mathcal{U})$ such that

$$F = p_X \circ H \quad \text{and} \quad G = p_Y \circ H$$